Characterization of Polynomials:-

Q No.→ Show that a function which has no Singularity in the finite Part of the Plane and has a Pole of order n at infinity is a Polynomial of degree n.

Proof:- Since $f(z)$ has no Singularity in the finite Part of the z-Plane, it can be expanded in a Taylor's Series about $z=0$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$\therefore f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} a_n z^{-n} \quad \text{--- (1)}$$

Again, it is given that $f(z)$ has a Pole of order n at infinity.

Therefore, $f\left(\frac{1}{z}\right)$ has a Pole of order n at zero.

$\therefore$ Laurent's expansion of $f\left(\frac{1}{z}\right)$ is valid in any region, which does not contain zero.

$$\therefore f\left(\frac{1}{z}\right) = F(z) + \sum_{m=1}^{n} B_m z^{-m} \quad \text{--- (2)}$$

From ① & ②, it follows that

$$\sum_{n=0}^{\infty} a_n z^{-n} = F(z) + \sum_{m=1}^{m} B_m z^{-m},$$

Equating, the coefficients of like power of both side it follows that

$a_0 = F(z) = \text{Constt}, \quad a_{n+1} = a_{n+2} = \text{------} = 0.$

Hence, $f(z) = a_0 + a_1 z + a_2 z^2 + \text{----} + a_n z^n$.

i.e. $f(z)$ is a Polynomial of degree n.

QNo→ A rational number has no Singularities other than Poles.

Proof:- Let $f(z)$ be a rational function.

∴ We can express $f(z)$ as,

$$f(z) = \frac{\psi(z)}{\phi(z)}, \text{ where } \phi(z) \neq 0$$

and $\psi(z)$ and $\phi(z)$ have no factor in Common.

Clearly the Singularity of $f(z)$ in the finite Part of the z-Plane are given by $\phi(z)=0$-

It is obvious that the zeros of $\phi(z)$ are the Pole of $\frac{1}{\phi(z)}$.

Hence, $f(z) = \frac{\psi(z)}{\phi(z)}$ Can have no other Singularity.

other then Poles in the finite Part of the z-Plane.