M.A./M.Sc. (Mathematics) Semester-I DR. SUNIL JEE
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Paper-I (Abstract Algebra)

Subgroup:-

(Defn.) Subgroup:- Let (G, o) be any group and H be any non-empty Sub set of G. If the set H together with the operation o forms a group (H, o). Then (H, o) is called a Sub-group of (G, o).

(Theorem 1). If H be a Sub-group of a group G, Prove that the identity of H is the same as that of G.

Proof:- Let e and e' be the identity element of G and H respectively.

Then, we have to Prove that $e' = e$.

Let $a \in H$. Then,

$e'a = a$. [by identity axiom.]

Since H is a Sub-group of the group G, therefore $a \in H \Rightarrow a \in G$.

$\Rightarrow ea = a$, [by identity axiom]

Hence, in G, we have

$e'a = ea$

$\Rightarrow e' = e$ [by right Cancellation law].

Hence the theorem is Completed.

(Theorem 2.) To Prove that the inverse of any element of a Sub-group is the Same as the inverse of that element regarded as an element of the group.

Proof:- Let H be a Sub-group of the group G. Then if e be the identity of H, e must be the identity of G. Let $a \in H$.

Since H is a Sub-group of G, therefore $a \in G$.

Let a^{-1} be the inverse of a and $a^{-1} \in H$.

If a' be the inverse of a and $a' \in G$. Then we have to Prove that $a^{-1} = a'$.

By inverse axiom, we get

$a^{-1}a = e$, taking Case of H.

and $a'a = e$, taking Case of G.

Hence in G, we have $a^{-1}a = a'a$

$\Rightarrow a^{-1} = a'$ [by right Cancellation law]

Hence the Complete theorem.

(Theorem 3):- A non-empty Sub-Set H of a group G is a Sub-group of G if and only if

(i) $a \in H, b \in H \Rightarrow ab \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$ where a^{-1} is the inverse of a in G.

Proof:- Necessary Condition:-

Let H be a Sub-group of the group G. Then H must Possess the Closure axiom, viz,

$a \in H, b \in H \Rightarrow ab \in H$

Hence the Part (i) is Proved.

Let $a \in H$. Then $a \in G$, as H is a Sub-group of G.

Let a^{-1} be the inverse of a in G.

Since H is itself of group, So a^{-1} must be an element of H.

Hence, $a \in H \Rightarrow a^{-1} \in H$.

Thus the Part (ii) is Proved.

So, the necessary Conditions are Satisfied.

The Sufficient Condition:-

Let the Conditions given by (i) and (ii) hold. Then We have to Prove that H is Sub-group of G.
By (i) $a \in H, b \in H \Rightarrow ab \in H$. Hence H is Closed with respect to multiplication operation.
Since G is a group therefore the operation in G is associative.
Also, Since H is Subset of G, therefore all the elements of H must belong to G. Hence it follows that the operation in H must also be associative.
Let e be the identity element in G.
Then, $aa^{-1} = e \ \forall \ a \in G$
Also, $a \in H \Rightarrow a^{-1} \in H$
$\Rightarrow aa^{-1} \in H$
$\Rightarrow e \in H.$ $[\because aa^{-1} = e]$
Hence e is the identity element of H.
By (ii) $a \in H \Rightarrow a^{-1} \in H$.
Hence each element of H Possesses an inverse.
We find that the Set H Satisfies all the axioms of group.
Hence H is a group. But H is a Sub-set of the group G.
So H is a Subgroup of G.
Thus, the Condition is Sufficient.