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QNo→ Every Compact set is bounded.

or QNo→ Show that a Compact set in a metric space is bounded.

Proof: Suppose that A is a Compact subset of metric space X. Suppose that A is not bounded. Then for a fixed point $x_0 \in X$, we can choose $x \in A$ such that $d(x_1, x_0) > 1$.

Then we choose $x_2 \in A$ such that

$$d(x_2, x_0) > d(x_1, x_0) + 1$$

Having chosen x_{n-1}, we choose $x_n \in A$ such that,

$$d(x_n, x_0) > d(x_1, x_0) + \cdots + d(x_{n-1}, x_0) + 1.$$

Thus for $n \geq m$, we have

$$d(x_n, x_0) \geq d(x_m, x_0) + 1.$$

Also, $d(x_n, x_0) \leq d(x_n, x_m) + d(x_m, x_0)$

Combining these we get $d(x_n, x_m) > 1$.

So we get a sequence $\{x_n\}$ which has no Convergent subsequence. This contradicts the Compactness of A.

Hence A is bounded.

(Theo^m):- Closed subset of Compact sets are Compact.

Proof:- Let F be a closed subset of a Compact set K in a metric space X. Let $\{G_\alpha\}$ be an open Cover of F. Since F is closed F^c is open. So $\{G_\alpha, F^c\}$ is an open Cover of K. The Compactness of K implies that this open Cover has a finite subcover, say $\{G_{\alpha_1}, \dots G_{\alpha_n}, F^c\}$. If we discard F^c from this subcover then the remaining subcover $\{G_{\alpha_1}, G_{\alpha_2}, \dots G_{\alpha_n}\}$ is a finite subcover of F Hence F is Compact.

(Theo^m) If E is an infinite subset of a Compact set K then E has a limit Point in K.

Proof:- If Possible, Suppose that no Point of K is a limit Point of E. Then with each $q \in K$, we can associate a n'hood V_q which Contains at most one Point of E. The Collection $\{V_q\}$ is an open Cover of K which has no finite subcover, because no finite subcollection Can Cover E (and hence K). This Contradicts the Compactness of K. Hence E has a limit point in K.

(Theorem) A Compact metric Space is Complete.

Proof:- Let X be a Compact metric Space and let $\{p_n\}$ be a Cauchy Sequence in X. Then for given $\varepsilon > 0$ there exists a Positive integer j such that $n > j \Rightarrow d(p_n, p_j) < \varepsilon$.

Since X is Compact, there is a subsequence $\{p_{n_k}\}$ such that $\lim_{k \to \infty} p_{n_k} = p$ (Say).
So there exists a positive integer $m > j$ such that, $d(p_{n_m}, p) < \varepsilon$.
Then, $n_m > m > j$ and So,

$$d(p_n, p) \leq d(p_n, p_j) + d(p_j, p_{n_m}) + d(p_{n_m}, p)$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

Hence, $p_n \to p$. This Proves that X is Compact.

(Theorem) A Compact Subset of a metric Space is Closed.

Proof:- Suppose that A is a Compact Subset of a metric Space X. If possible, Suppose that A is not Closed. Then it has a limit Point $p \notin A$. So we can choose a Sequence $\{p_n\}$ in A which Converges to p. Since $\{p_n\}$ is a Convergent Sequence every subsequence of it also Converges to $p \notin A$. This Contradicts the Compactness of A. Hence A is closed.

(Theorem) Every sequence in a Compact metric space has a convergent subsequence.

Proof:- Let $\{x_n\}$ be a sequence in a compact metric space X. Let E be the range of $\{x_n\}$. If E is finite, then there is a sequence $\{n_k\}$ of positive integers with $n_1 < n_2 < n_3$ ------ such that

$$x_{n_1} = x_{n_2} = \text{-----} = x_{n_k} \text{------} = x \in E.$$

Thus, the subsequence $\{x_{n_k}\}$ converges to x.

Suppose that E is infinite. Then E has a limit point in X, say x. We choose n_1 such that $d(x_{n_1}, x) < 1$. Then we choose $n_2 > n_1$ such that $d(x_{n_2}, x) < \frac{1}{2}$. Having choosen n_{k-1}, we choose $n_k > n_{k-1}$ such that $d(x_{n_k}, x) < \frac{1}{k}$. The subsequence $\{x_{n_k}\}$. So constructed converges to x.

Hence every sequence has a convergent subsequence in a compact metric space.