(iii) Linear Differential Equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad ...(i)$$

Where P and Q are functions of x or constants is called a linear differential equation.

To solve Eq. (i), we multiply both sides of Eq. (i) by $e^{\int P dx}$ (called integrating factor) so that Eq. (i) reduces to

$$\frac{d}{dx}\left[y \cdot e^{\int P dx}\right] = Q \cdot e^{\int P dx}$$

On integrating both sides, we get

$$y \cdot e^{\int P dx} = \int \left(Q \cdot e^{\int P dx}\right) dx + C$$

$\Rightarrow$

$$y \times \text{IF} = \int (Q \times \text{IF})\, dx + C$$

where

$$\text{IF} = e^{\int P dx}$$

Example 10. *Solve* $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^2(2x-1)$

Solution. We have $\dfrac{dy}{dx} - \dfrac{x-2}{x(x-1)}y = \dfrac{x^2(2x-1)}{x(x-1)}$

Hence, the integrating factor is

$$\text{IF} = e^{-\int \frac{x-2}{x(x-1)}dx} = e^{\int \left(\frac{1}{x-1} - \frac{2}{x}\right)dx}$$

$$= e^{\log(x-1) - 2\log x} = e^{\log \frac{x-1}{x^2}} = \frac{x-1}{x^2}$$

Hence, the solution is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x(x-1)} \cdot \frac{x-1}{x^2} dx + C = \int \frac{2x-1}{x} dx + C$$

$$= \int \left(2 - \frac{1}{x}\right) dx + C = 2x - \log x + C$$

Example 11. *If* $\dfrac{dy}{dx} + 2y \tan x = \sin x$ *and* $y\left(\dfrac{\pi}{3}\right) = 0$, *then find maximum value of* $y(x)$.

Solution. We have $\dfrac{dy}{dx}+2y\tan x=\sin x$

$$\text{IF}=e^{2\int \tan x\,dx}=e^{2\log \sec x}=e^{\log \sec^2 x}=\sec^2 x$$

Hence, solution is

$$y\cdot \sec^2 x=\int \sec^2 x\cdot \sin x\,dx+C=\int \tan x\sec x\,dx+C$$

$$=\sec x+C$$

Put $x=\dfrac{\pi}{3}, y=0$ in Eq. (i),

$$0\times \sec^2\frac{\pi}{3}=\sec\frac{\pi}{3}+C$$

$$0=2+C \text{ or } C=-2$$

Substituting $C=-2$ in Eq. (i), we get

$$y\sec^2 x=\sec x-2$$

$$y=\cos x-2\cos^2 x,$$

$$\frac{dy}{dx}=-\sin x+4\cos x\sin x$$

For maxima or minima, we have

$$0=-\sin x+4\sin x\cos x$$

$$\Rightarrow \quad -\sin x(1-4\cos x)=0$$

$$\Rightarrow \quad \sin x=0 \text{ or } \cos x=\frac{1}{4}$$

Now, $\dfrac{d^2y}{dx^2}=-\cos x-4\sin^2 x+4\cos^2 x$

If $\cos x=\dfrac{1}{4}$, then

$$\frac{d^2y}{dx^2}=-\frac{1}{4}-4\sqrt{1-\frac{1}{16}}+4\left(\frac{1}{16}\right)=-\sqrt{15}$$

So, y is maximum if $\cos x=\dfrac{1}{4}$, Maximum value of

$$y=\frac{1}{4}-\frac{2}{16}=\frac{1}{8}$$

Example 12. *Solve* $x^2dy+y(x+y)dx=0$

Solution. We have, $\quad x^2dy+y(x+y)dx=0$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x}=-\frac{y^2}{x^2} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=-\frac{1}{x^2}$$

Put $\quad -\dfrac{1}{y}=z \quad$ so that $\quad \dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{dz}{dx}$

... linear differential equation